

Weak Decoherence and Quantum Trajectory Graphs

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Abstract

Griffiths' "quantum trajectories" formalism is extended to describe weak decoherence. The decoherence conditions are shown to severely limit the complexity of histories composed of fine-grained events.

In response to the increasingly popular opinion that the Copenhagen interpretation of quantum mechanics raises more questions than it answers [7] and a desire to treat the entire universe quantum mechanically, Gell-Mann and Hartle [1, 2, 3] have worked to create an alternative interpretation of quantum theory, expanding upon earlier work by Griffiths [4] and Omnès [6]. Their scheme emphasizes not individual events but Griffiths' notion of a *history*, a sequence of events at a succession of times, and they assert that the

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histories to which one assigns probabilities are distinguished not by measurements made by an external classical “observer” but by the extent to which they satisfy certain “consistency” or “decoherence” conditions guaranteeing compliance with the classical rules of probability.

Yet basic questions remain largely unanswered: How restrictive are the decoherence conditions? What kinds of histories decohere? Do they occur in sufficient variety to describe the physical world?

These questions have led us to investigate several aspects of decoherence. We have extended Griffiths’ “quantum trajectories” formalism [5] to describe *weakly* decohering sets of histories. We have found severe limits on the structure of fine-grained decohering histories.

Following Gell-Mann and Hartle, we let an *event* be described by a projection operator P_α . If P_α is one-dimensional we say the event is *fine-grained*; otherwise the event is *coarse-grained*. A *complete* set of events $\{P_\alpha\}$ forms a resolution of the identity:

$$\sum_{\alpha} P_{\alpha} = I \quad \text{and} \quad P_{\alpha} P_{\beta} = \delta_{\alpha\beta} P_{\beta}. \quad (1)$$

Let $\{P_{\alpha_k}(t_k)\}$ be the complete set of events (in the Heisenberg picture) at time t_k . The probability that event $P_{\alpha_1}(t_1)$ will occur at time t_1 , $P_{\alpha_2}(t_2)$ at time t_2 , \dots , and $P_{\alpha_n}(t_n)$ at time t_n is [2]

$$p(\alpha_1, \alpha_2, \dots, \alpha_n) = \text{Tr} (P_{\alpha_n}(t_n) \dots P_{\alpha_2}(t_2) P_{\alpha_1}(t_1) \rho P_{\alpha_1}(t_1) P_{\alpha_2}(t_2) \dots P_{\alpha_n}(t_n)) \quad (2)$$

for an initial state described by a density operator ρ . With this sequence of events we associate the *history* C_α defined by

$$C_\alpha = P_{\alpha_n}(t_n) \dots P_{\alpha_2}(t_2) P_{\alpha_1}(t_1), \quad (3)$$

in terms of which (2) becomes

$$p(\alpha) = \text{Tr} (C_\alpha \rho C_\alpha^\dagger). \quad (4)$$

With this expression in mind, we define the *decoherence functional* between histories C_α and C_β for an initial state ρ by

$$D(\alpha, \beta) = \text{Tr} (C_\alpha \rho C_\beta^\dagger), \quad (5)$$

and we say $\{C_\alpha\}$ forms a *weakly decohering* set of histories iff

$$\text{Re } D(\alpha, \beta) = 0 \text{ for all } \alpha \neq \beta. \quad (6)$$

This condition guarantees that the probabilities associated with the histories in $\{C_\alpha\}$ obey the classical rules of probability [2]. For such a set of histories, the decoherence condition (6) and the probability formula (4) may be combined in the equation

$$\text{Re } D(\alpha, \beta) = p(\alpha)\delta_{\alpha\beta}. \quad (7)$$

A set of histories which satisfies the stronger condition

$$D(\alpha, \beta) = p(\alpha)\delta_{\alpha\beta} \quad (8)$$

is said to exhibit *medium decoherence*. This condition is sufficient but not necessary to ensure compliance with the classical rules of probability. We will consider both kinds of decoherence. Finally, we will speak of individual histories C_α and C_β decohering if they satisfy (7) (or (8)).

Histories with different final events always decohere and any history which occurs with zero probability decoheres with all other histories. Further, any decohering set of histories can be extended by inserting between any two times a set of events identical (in the Heisenberg picture) to those at the earlier or later time; the set of histories that results still decoheres. This corresponds to inserting a set of events in the Schrodinger picture which matches the earlier or later set aside from unitary evolution to the new time. We call this a *congruent extension*, since the new events are congruent with the old ones. In light of this, we will look for sets of histories with more than one nonzero-probability history but without congruent extensions.

We will use Griffiths' graphical representation [5] of "consistent histories," in which he represents the set of possible events at each time with an orthonormal basis of the system's state space. (In this formalism, every event is fine-grained.) He represents the set of histories produced by this choice of events with a *trajectory graph*, in which each event at time t_j corresponds to a node in the j^{th} column of the graph and a line is drawn between nodes in adjacent columns iff the transition amplitude between the corresponding events is nonzero. Figure 1 presents two examples of such graphs. Each path (unbroken line through two or more nodes) through a trajectory graph

represents a nonzero-probability history with initial state given by the first node in the path. The set of histories described by the graph satisfies the *noninterference condition* if any two nodes are connected by at most one path. We will show immediately below that the noninterference condition is equivalent to *medium* decoherence of the set of histories with any node in the graph as the initial state. However, we wish to follow in the spirit of Gell-Mann and Hartle, in which decoherence is a function of the initial state as well as the histories themselves. Further, since medium decoherence is a more stringent requirement than is actually necessary, we would like to have a condition for *weak* decoherence in terms of these graphs. As we will also prove below, the required condition is that *at most two distinct paths connect any two events, and if there are two paths, the phases of the corresponding amplitudes differ by $\frac{\pi}{2}$* . Thus, we will use a modified form of Griffiths' quantum trajectory formalism in which (1) we specify the initial state (producing what Griffiths would call an *elementary family* of trajectories) and (2) we impose the requirement of weak, not medium, decoherence. Figures 2 and 3 provide examples of such graphs.

Both decoherence conditions mentioned above are special cases of the following theorem.

Theorem 1. Suppose $\{C_\alpha\}$ is a decohering set of histories with initial state ρ and suppose that two possible events $|j\rangle\langle j|$ at time t_j and $|k\rangle\langle k|$ at a later time t_k are fine-grained. If at least one history leading to event $|j\rangle\langle j|$ occurs with nonzero probability, then of all histories which lead from $|j\rangle\langle j|$ to $|k\rangle\langle k|$, at most *two* occur with nonzero probability. If two occur, then the phases of the corresponding amplitudes differ by $\frac{\pi}{2}$. If the set $\{C_\alpha\}$ exhibits *medium* decoherence, at most *one* history leading from $|j\rangle\langle j|$ to $|k\rangle\langle k|$ occurs with nonzero probability.

Proof. Any history leading from $|j\rangle\langle j|$ to $|k\rangle\langle k|$ can be written as

$$C_\alpha = |k\rangle\langle k|D_\alpha|j\rangle\langle j| \quad (9)$$

and the decoherence condition (6) applied to any two histories which include C_α and C_β where $\alpha \neq \beta$ can be reduced to

$$\text{Re} \langle k|D_\alpha|j\rangle\langle k|D_\beta|j\rangle^* = 0. \quad (10)$$

If both amplitudes are nonvanishing, then

$$\arg(\langle k|D_\alpha|j\rangle) - \arg(\langle k|D_\beta|j\rangle) = \pm\frac{\pi}{2}. \quad (11)$$

Thus, any two numbers in the set $\{\langle k|D_\alpha|j\rangle\}$ are orthogonal in the complex plane. Since the complex plane is *two-dimensional*, at most *two* of the $\langle k|D_\alpha|j\rangle$ are nonzero. If there are two, they have the promised phase difference of $\frac{\pi}{2}$. Had we assumed that the histories exhibited *medium* decoherence, we would have used the decoherence condition (8) and in (10) we would not have taken the real part; then at most *one* member of the set $\{\langle k|D_\alpha|j\rangle\}$ would be nonvanishing. \square

If the initial state of the system is pure, then the theorem is still valid if we replace $|j\rangle\langle j|$ with ρ . Thus, if the initial state is pure and the set $\{C_\alpha\}$ exhibits weak (medium) decoherence, then at most two histories connect (one history connects) the initial state to any fine-grained event with nonzero probability.

An immediate consequence of this theorem is interesting enough to be a theorem of its own.

Theorem 2. If $t_j < t_k < t_l$ and a nonzero-probability history leads to a fine-grained event at t_j which does not occur at t_k but occurs again at t_l , then no set of histories containing these events can decohere.

Proof. At least two nonzero-probability histories must connect the event at t_j to its twin at t_l . Further, the product of the amplitude for one history and the complex conjugate of that for the other is real (and positive), because the factors linking t_j to t_k are the complex conjugates of those linking t_k to t_l . Thus condition (10) of Theorem 1 cannot be satisfied. \square

With Theorem 1 in hand we can immediately describe all possible decohering sets of histories of a two-level system (spin $\frac{1}{2}$) with a pure initial state. All sets of histories with one event after the initial state exhibit (medium) decoherence automatically; thus we begin by considering two-event sets. We assume the system is initially polarized in the direction \vec{r} , polarized parallel or antiparallel to \vec{n} at t_1 , and parallel or antiparallel to \vec{f} at t_2 . Writing the corresponding projection operators in the standard way using Pauli matrices,

one discovers that the weak decoherence condition (6) becomes

$$(\vec{i} \times \vec{n}) \cdot (\vec{n} \times \vec{f}) = 0. \quad (12)$$

(This result is not new [7].) Only these sets of two-event histories weakly decohere. Further, every decohering set of histories with three or more events is a congruent extension of a two-event set; if it were not, the number of nonzero-probability histories would be at least five, so at least three would lead from the initial state to one of the two final events, which Theorem 1 does not allow.

If we were to impose *medium* decoherence instead, the allowed sets of histories would simplify considerably. Theorem 1 allows at most one nonzero-probability history to lead from the initial to each of the final states; thus the total number of nonzero-probability histories would be at most two. Any set of histories which is not a congruent extension of a one-event set will have at least three nonzero-probability histories; thus it would not decohere. Therefore the only sets of fine-grained histories of a two-level system which exhibit medium decoherence are one-event sets and their congruent extensions. In Griffiths' language, we have shown that weakly decohering sets of histories corresponding to the graph in Figure 2(a) exist, but the only sets exhibiting medium decoherence are represented by graphs like the one in Figure 2(b), a congruent extension of a one-event set.

We call an event in a trajectory graph *connected* if its node leads back to the initial state through at least one path (if at least one history leading to the event from the initial state occurs with nonzero probability). We call it *singly* connected if exactly one path leads back to the initial state, *doubly* connected if two paths lead back to the initial state. In these terms, Theorem 1 demands that every fine-grained event in a decohering set of histories be at most doubly connected (or singly connected if the set exhibits medium decoherence). An event is *unconnected* iff it has no overlap with the connected events at the previous time; thus the unconnected events at any time lie in the span of the unconnected events at the previous time. Therefore the number of *connected* events is a nondecreasing function of time.

Theorem 3. In every transition between times in a decohering set of histories represented by a trajectory graph, either

1. the connected events before and after the transition are identical;

2. the number of connected events increases by at least one;
3. the number of doubly connected events increases by at least *two*; or
4. both 2 and 3 occur.

Proof. All we need to prove is that if 1 and 2 do not occur, then 3 must occur. Thus, suppose the connected events before and after the transition from time t_j to time t_{j+1} are not identical, yet the number of connected events does not increase. Then at least one event at t_{j+1} must be connected to two events at t_j , as shown in Figure 1(a). However, that one event at t_{j+1} cannot be the *only* one linked to two events at t_j ; since the first event at t_j is connected to only the first event at t_{j+1} , the two differ at most by a phase, and because the first and second events at t_j are orthogonal, the first event at t_{j+1} and the second event at t_j must also be orthogonal. Thus at least *two* events at t_{j+1} must be connected to two events (each) at t_j , as shown in Figure 1(b). None of the doubly connected events from t_j can be involved in this part of the transition (since that would make one of the events at t_{j+1} at least triply connected); therefore, the number of doubly connected events increases by at least two. \square

In a set of histories represented by a trajectory graph, the system's behavior is specified at only a finite number of times. We might have hoped to better approximate continuous time evolution by inserting additional sets of events between those already in the graph. However, as the next theorem shows, the possibilities for this are very limited.

Theorem 4. Suppose that between times t_j and t_{j+1} in a decohering set of histories represented by a trajectory graph, exactly one step of change occurs: either the number of connected events increases by *exactly* one or the number of doubly connected events increases by *exactly* two (but not both). Then if an additional set of events is inserted between t_j and t_{j+1} while maintaining decoherence, it must be identical to either the set at t_j or the set at t_{j+1} .

Proof. Suppose that the new set is identical to neither the set before nor the set after. Then in the transition from t_j to t_{j+1} at least two steps of change must occur (one for the transition from t_j to the intermediate time, one for the transition from the intermediate time to t_{j+1}). \square

The histories are restricted even more drastically if only *finitely* many events occur with nonzero probability (so only that many events are connected).

Theorem 5. Consider a trajectory graph representing a set of decohering histories with a finite number n of connected events at a particular time. Excluding congruent extensions, the number of transitions prior to that time is at most $n + \lfloor \frac{n}{2} \rfloor - 2$, where $\lfloor \cdot \rfloor$ denotes the greatest integer part.

Proof. Suppose that the given set of decohering histories contains no congruent extensions. The number of connected events at time t_1 is therefore at least two, so the number of transitions that increase the number of connected events is at most $n - 2$. The number of transitions that increase the number of doubly connected events is at most $\lfloor \frac{n}{2} \rfloor$. Thus the total number of transitions is at most $n + \lfloor \frac{n}{2} \rfloor - 2$. \square

This bound is the strongest possible, because for every n there is a set of decohering histories in an n -dimensional space with this maximum number of noncongruent steps (the $n = 5$ case is illustrated in Figure 3). The consequences of this theorem are avoided only if the number of connected events is infinite right at the start, so that infinitely many events occur with nonzero probability at each time.

Comparison with Figure 2 shows that in each of the last two transitions in Figure 3 the system can be decomposed into two subspaces, in one of which the transition is to congruent events while in the other the transition is that of a two-level system. In fact, a large class of transitions is of this general type, as we show with our final theorem.

Theorem 6. In every transition in which the number of connected events is finite and does not increase, the matrix describing the transition between the *connected* events is block-diagonal (to within rearrangement of the rows and columns), and each block is either 2×2 or 1×1 .

Proof. Let the transition from t_j to t_{j+1} leave the number of connected events n unchanged. Since the span of the connected events at t_j lies in the span of the connected events at t_{j+1} and both have dimension n , the two subspaces are the same; so they have the same orthogonal complement. Thus each (un)connected event at t_{j+1} overlaps *only* the (un)connected events at

t_j . Since each connected event at t_{j+1} is at most doubly connected, each is linked to either one or two connected events at t_j and no others; thus the matrix describing the entire transition has at most two nonzero entries in each column representing a connected event at t_{j+1} . If a column has only one nonzero entry, then unitarity guarantees that the entry is also the only nonzero entry in its row; this yields all of the 1×1 blocks. If a column has two nonzero entries, then the orthogonality of different rows and columns demands that the entries in the same two rows of *one* and *only* one other column are also nonzero. Those two columns together form a 2×2 block; all other entries in their rows and columns are zero. \square

This theorem reduces the allowed transitions to an extremely simple form; its restrictions are avoided only if the number of connected events (the number of events that occur with nonzero probability) increases continually over time.

These results suggest that the decoherence conditions strongly favor histories dominated overwhelmingly by congruent extensions. It is not surprising that decoherence selects out the histories that conform with the system's unitary evolution, but the extent to which they are preferred is remarkable. For example, only congruent events can occur between congruent events (Theorem 2), and if continuous classical evolution is to be approached by inserting events at more and more times, almost all insertions must be congruent extensions (Theorems 2, 4, and 5). Probabilities that are periodic in time and are for a finite number of events at some time must be for congruent events and are therefore constant in time. (The number of connected events can not decrease, so if it is periodic it must be constant and Theorem 6 applies.)

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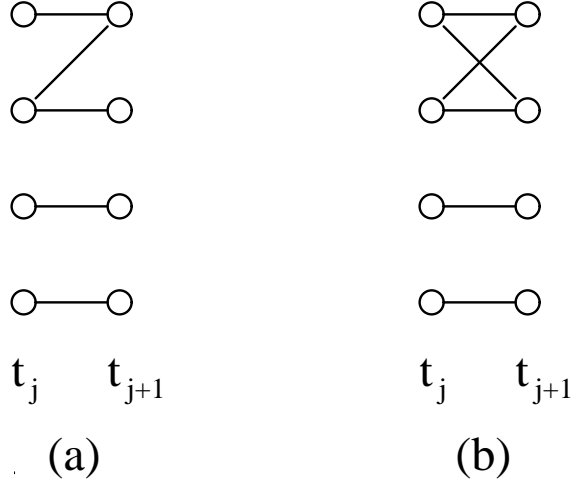


Figure 1: Griffiths trajectory graphs. (a) A candidate for a nontrivial transition in which the number of connected events does not increase. This graph is forbidden by the orthogonality of different events at t_j . (b) Another candidate for the same transition. The orthogonality of different events at t_j demands that at least two events at t_{j+1} be doubly connected.

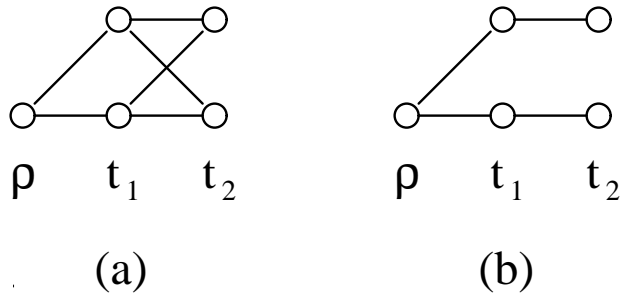


Figure 2: Trajectory graphs with a specified initial state. (a) A graph corresponding to a weakly decohering set of histories of a two-level system. (b) A graph corresponding to a set of histories exhibiting medium decoherence.

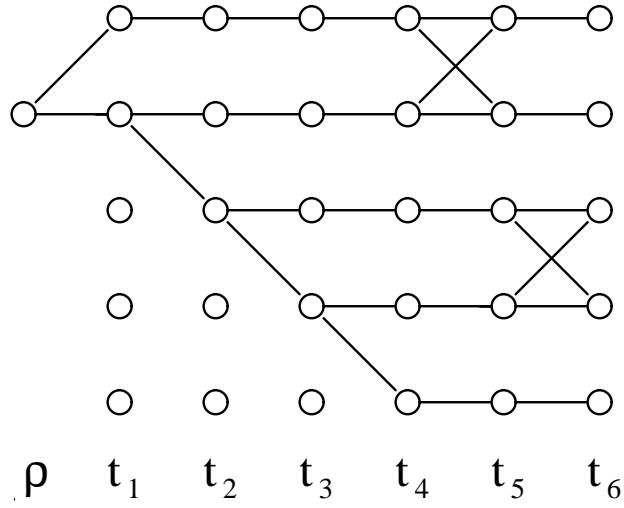


Figure 3: A set of decohering histories in a five-dimensional space with exactly $5 + \left\lceil \frac{5}{2} \right\rceil - 2 = 5$ noncongruent transitions.